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$Z(2)$ vortex solution in a field theory

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ABSTRACT

We present a finite energy topological $Z(2)$ vortex solution in a $2+1$ dimensional $SO(3)$ gauge field theory minimally coupled to a matrix valued Higgs field. The vortex carries a $Z(2)$ magnetic charge and obeys a modulo two addition property. The core of this vortex has a structure similar to that of the Abrikosov vortex appearing in a type II superconductor. The implications of this solution for Wilson loops are quite interesting. In two Euclidean dimensions these vortices are instantons and a dilute gas of such vortices disorders Wilson loops producing an area law behaviour with an exponentially small string tension. In $2+1$ dimensions the vortices are loops and they affect the same disordering in the phase having large loops.

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Vortices have played an important role in statistical mechanics and field theories. They are global excitations which arise because of the topology of the dynamical variables. Vortices play an important role in driving the phase transition in the two dimensional planar model [1]. They also appear as a result of many body interactions in a Type II superconductor [2]. An example of a vortex solution in a quantum field theory was given in [3]. This solution is the relativistic version of the Abrikosov vortex in the Type II superconductor. These vortices have an additive integer charge associated with them. Vortices were defined for non-abelain gauge theories in [4]. In this definition the vortex has a $Z(N)$ charge. The effect of such a vortex is to produce $Z(N)$ phase factors for Wilson loops surrounding the vortex. Vortices are just one of the many topological solutions that occur in field theories. Other well known examples of topological solutions are, the 't Hooft-Polyakov monopole [5], the instanton [6], and the skyrmion [7]. These solutions have played an important role in uncovering the non-perturbative effects in quantum field theories. Since they are global effects they have consequences which cannot be seen in the usual perturbative expansion. The 't Hooft-Polyakov monopole solution was used in [8] to study the Higgs phase of the Georgi-Glashow model and it was shown it radically alters the naive spectrum of the theory. Instanton solutions have been used with varying success [9] to understand the long distance properties of QCD. Skyrmion solutions [7] also lead to important non-perturbative effects. The vortex solutions considered so far were abelian vortices, they have an integral vortex charge. We present here an example of a $Z(2)$ vortex solution and then examine the consequences of this solution for the field theories in which it occurs.

Since we will be interested in vortex solutions we mention some general properties of vortices, first in three and then in four space-time dimensions. In three dimensions a $Z(2)$ vortex is said to pierce a two dimensional region (simply connected) R if every Wilson loop surrounding this region picks up a phase ($Z(N)$ for the group $SU(N)$). The effect of this vortex can also be understood as the action of a pure gauge transormation which is not single valued on every closed loop surrounding R. Because of topological

considerations the region R cannot be arbitrarily shrunk by regular gauge transformations and this is the region associated with the core of a vortex. If the vortex solution is to have a finite energy the region outside the core must have a zero energy density. The existence of vortex solutions is determined by the first homotopy group, $\pi_1(H)$, of the field configurations in the region oustide the vortex (H being the field configuration space outside the vortex). The vortex can extend in the dimensions orthogonal to R and can either stretch indefinitely, form closed loops, or end in objects(monopoles) which absorb the vortex flux. In four dimensions the above picture gets repeated on every slice in the extra dimension and the vortex line becomes a vortex sheet whose area is the product of the length of the vortex and the duration in time for which the vortex propagates. These vortex sheets can form closed two dimensional surfaces or they can end in monopole loops. A famous example of a vortex solution in a gauge theory is the Nielsen-Olesen vortex which occurs in three dimensional scalar QED [3]. This vortex carries an integral magnetic flux (because $\pi_1(U(1)) = Z$), and it is a relativistic generalization of the Abrikosov vortex appearing in a Type II superconductor. Apart from the Abrikosov-Nielsen-Olesen vortex there are not many other vortex solutions known. We present here an example of a vortex solution where the vorticity can have only one non-trivial value. This solution occurs in an $SO(3)$ invariant gauge theory coupled to a Higgs field.

The field theory under consideration is an $SO(3)$ gauge invariant theory minimally coupled to a matrix valued Higgs field M in the $(3,3)$ representation of $SO(3)$. The Lagrangian (in $2+1$ space-time dimensions) is given by

$$L = \int d^2x \ d\tau \frac{1}{2} tr((D_\mu M)^t (D_\mu M)) - \frac{1}{4} F^\alpha_{\mu\nu} F^\alpha_{\mu\nu} - trV((M^t M)) \quad . \tag{1}$$

$D_\mu M$ is the covariant derivative and is defined in the usual way as $D_\mu M = \partial_\mu M + g[A_\mu, M]$. The gauge field A_μ can be written in terms of its Lie Algebra valued components as $A_\mu = A^\alpha_\mu T^\alpha$. T^α are $3X3$ matrices in the Lie algebra of $SO(3)$ and satisfy $[T^\alpha, T^\beta] = \epsilon^{\alpha\beta\gamma} T^\gamma$ (note that since we are dealing with $SO(3)$ there is no i factor). Under local $SO(3)$

transformations ($V(x)$) the fields transform as

$$
\begin{aligned}
M &\to VMV^{-1} \\
A_\mu &\to VA_\mu V^{-1} - \frac{1}{g}(\partial_\mu V)V^{-1} \quad .
\end{aligned}
$$

The equations of motion are given by

$$
\begin{aligned}
D_\mu F_{\mu\nu\alpha} &= -g\ tr((D_\nu M)^t [T^\alpha_\nu, M]) \\
\partial_\mu (D_\mu M) &= -[A_\mu, D_\mu M] - \frac{\partial V}{\partial M} \quad .
\end{aligned}
$$

(There is no summation over the ν index in the rhs of the first equation). We will be interested in classical vortex solutions of these equations of motion. We mimic the approach taken in [3] to look for these solutions. Outside the core of the vortex we require the fields to have zero energy density so that the solution has a finite energy. A set of such zero energy density configurations outside the core must satisfy

$$
\begin{aligned}
D_\mu M &= 0 \\
F^\alpha_{\mu\nu} &= 0 \\
V(M) &= 0 \\
\frac{\partial V}{\partial M} &= 0 \quad .
\end{aligned}
$$

The last two conditions ensure that the potential energy density (associated with the field M) is zero outside the vortex core and that the field M is always at the minimum of the potential energy (also taken to be zero). A field obeying these conditions is said to be in a Higgs vacuum. A vortex solution is one which winds around in a non-trivial way in some internal space for a closed loop traversal in physical space. It is clear that the existence of such configurations is determined by the structure of the first homotopy group of the Higgs vacuum. The Abrikosov-Nielsen-Olesen vortex is an example of a realization of a non-trivial element the first Homotopy group of the group $U(1)$. The relevant group in our example is $SO(3)$. The Higgs vacuum can be chosen in such a way that it consists of

elements of SO(3). One choice of Higgs potential that accomplishes this is

$$V(M) = \lambda(M^t M - I)^2 \quad . \tag{2}$$

The minimum of this potential is the set of orthogonal matrices

$$M^t M = I \quad . \tag{3}$$

Any choice of the vacuum M satisfying the above condition breaks the $SO(3)$ gauge symmetry. However, a specific constant choice of M, say M_s, is still invariant under a subgroup of $SO(3)$ gauge transformations such that

$$[R, M_s] = 0 \quad . \tag{4}$$

Thus there is still an unbroken gauge symmetry corresponding to the $SO(3)$ matrices which commute with M_s. The set of such matrices form a subgroup $SO(2)$ of $SO(3)$. If the matrix M_s is chosen to be

$$M_s = \exp(T^3 \alpha) \tag{5}$$

for some α, then gauge transformations about the T^3 axis will leave the vacuum invariant. The field A^3_μ will be the massless $SO(2)$ gauge field. Excitations about M_s can be described by a gauge theory (with gauge group $SO(2)$) in which there are two oppositely charged massive gauge bosons, one massless boson, and a scalar field. However, we are not interested in the excitations about the conventional Higgs phase. We are looking for global solutions which, even though they belong to the Higgs vacuum, cannot be obtained continuously (to any finite order in perturbation theory) from the conventional Higgs phase. We look for static solutions having zero energy density outside the core of the vortex. We also impose the following conditions

$$A^\alpha_0 = 0$$

$$A^1_\mu = A^2_\mu = 0 \quad .$$

The first of the above conditions, along with the static condition, implies a zero electric field. In addition to these conditions, only the 3 component (of the Lie algebra) of the gauge field is taken to be non-zero and this is chosen to have the same form as in the A-N-O vortex

$$\begin{aligned} A_1^3(x) &= \frac{-y}{gr^2} \\ A_2^3(x) &= \frac{x}{gr^2} \quad . \end{aligned}$$

In polar co-ordinates the solution has the form

$$\begin{aligned} A_\theta^3 &= \frac{1}{gr} \\ A_r^3 &= 0 \quad . \end{aligned}$$

The above solution satisfies $F_{12}^3 = 0$. We note for later reference that any A_θ^3 proportional to $\frac{1}{r}$ is also a solution. The field equation for M is satisfied if we take M to be a independent of r and $M^t M = I$. This leads to the following equation for M

$$\frac{1}{r}\frac{\partial M}{\partial \theta} + g[A_\theta, M] = 0 \quad . \tag{6}$$

Since only the 3 component of A_μ is non-zero the above equation reduces to

$$\frac{1}{r}\frac{\partial M}{\partial \theta} + gA_\theta^3[T^3, M] = 0 \quad . \tag{7}$$

The above solution can be solved easily by noting that for small $\delta\theta$,

$$M(\theta + \delta\theta) = \exp(-T^3\delta\theta)M(\theta)\exp(T^3\delta\theta) \tag{8}$$

solves Eq. 7 for $A_\theta^3 = \frac{1}{gr}$. This leads to the following solution

$$M_0(\theta) = \exp(-T^3\theta)M(0)\exp(T^3\theta) \quad . \tag{9}$$

The gauge field for this solution will be labelled as $A_{\theta\ 0}^3$. However, there is another solution of the form

$$M_1(\theta) = \exp(-T^3\frac{\theta}{2})M(0)\exp(T^3\frac{\theta}{2}) \tag{10}$$

that also solves Eq. 7 provided $A_\theta^3 = \frac{1}{2gr}$. This gauge field will be labelled as $A_{\theta\ 1}^3 = \frac{1}{2gr}$. We will show that the two solutions,M_0 and M_1, belong to different elements of the first Homotopy group of $SO(3)$ ($\Pi_1(SO(3)) = Z(2)$). To see this we need a particular parametrization of the elements of the group $SO(3)$. An element of $SO(3)$ (R) can be parametrized as

$$R(n,\tilde{\theta}) = \exp(T^\alpha n^\alpha \tilde{\theta}) \quad , \tag{11}$$

where n is a unit vector which takes values on the surface of a two dimensional sphere. $\tilde{\theta}$ varies in the range $0 \leq \tilde{\theta} \leq \pi$. Every $SO(3)$ element is parametrized by an axis of rotation n and an angle of rotation $\tilde{\theta}$. The $SO(3)$ manifold is thus the solid ball of radius π. Since a rotation around an axis n by an angle π is the same as the rotation around the axis $-n$ by the same angle π, antipodal points on the surface of the ball are identified. It is this identification that gives a non-trivial topology to the $SO(3)$ group. The matrix R can be written out as

$$R(n,\tilde{\theta}) = I + (T^\alpha n^\alpha)\sin(\tilde{\theta}) + (T^\alpha n^\alpha)^2(1-\cos(\tilde{\theta})) \quad . \tag{12}$$

(We have used the property $(T^\alpha)^3 = -T^\alpha$, expanded the exponential, and reordered the various terms.) The matrices T^α are the antisymmetric $3X3$ matrices belonging to the Lie algebra of $SO(3)$. Of the two solutions presented earlier, one is topologically trivial, the other is topologically non-trivial and this is the one that will be referred to as the vortex. The solutions presented are just one of a class of many possible solutions. In the above solutions if we choose $M(0) = \exp T^2\pi$, it is easy to see that $M_1(\theta)$ moves along the equator of the ball and returns to itself whereas $M_2(\theta)$ goes to the point diametrically opposite to M_0. This follows from the property

$$\exp(T^3\theta)\exp(T^2\pi)\exp(-T^3\theta) = \exp(T^\alpha n^\alpha \pi) \tag{13}$$

where $n^1 = -\sin\theta \;\; n^2 = \cos\theta \;\; n^3 = 0$. The way of visualizing this is that the point on the surface of the ball $\tilde{\theta} = \pi, n^1 = n^3 = 0, n^2 = 1$ gets rotated in the equatorial plane of the ball

($n^1 - n^2$ *plane*) by an angle θ. In the solution M_0, the vector n rotates by 2π whereas in the solution M_1 the vector n rotates by π, and the two paths belong to different elements of $\Pi_1(SO(3))$. Since opposite points are identified only on the surface of the ball, it is crucial to choose $M(0)$ to lie on the surface of the ball in the solution $M_1(\theta)$. In $M_0(\theta)$ the point $M(0)$ can be chosen anywhere, although it has also been chosen to be on the surface of the ball, because this is the solution which is topologically trivial. The two solutions can be transformed into each other by the following singular gauge transformation

$$V_s(\theta) = \exp(-T^3 \frac{\theta}{2}) \quad . \tag{14}$$

This follows from the relation

$$M_0(\theta) = V_s(\theta) M_1(\theta) V_s^{-1} \quad . \tag{15}$$

This gauge transformation takes $A^3_{\theta\ 1} = \frac{1}{2gr}$ to $A^3_{\theta\ 0} = \frac{1}{gr}$. The above gauge transformation has a discontinuity on the $\theta = 0$ axis because

$$V_s(2\pi) = \exp(-T^3 \pi) \neq V_s(0) \quad . \tag{16}$$

($TrV_s(2\pi) = -1,\ TrV_s(0) = 3$). An interesting feature is that two topologically non-trivial solutions annihilate each other because

$$A_{\theta\ 0} = A_{\theta\ 1} + A_{\theta\ 1} \quad , \tag{17}$$

or, in terms of the M fields,

$$M_1(\theta)^2 = I \quad . \tag{18}$$

Therefore, these vortices have a $Z(2)$ magnetic charge. It is worth pointing out that the $Z(2)$ magnetic charge that arises here has nothing to do with the original gauge group $SO(3)$. Unlike $SU(2)$ which has $Z(2)$ as its center, the group $SO(3)$ has no non-trivial center subgroup. We note that the solution $M_0(\theta)$ can be completely gauge transformed away and corresponds to the zero field case (and hence zero energy) case. The above

solutions are valid only outside the vortex core; in order to have a finite energy non-singular solution inside the vortex core we must look for solutions which are not singular at the core and which continuously go over to the solutions outside the core. We look for solutions inside the core by writing

$$A^3_\theta(r) = A^3_{\theta 1}(r) + A^3_{\theta c}(r),$$
$$M(r,\theta) = M_1(\theta) M_c(r) \quad .$$

It is assumed above that the angular behaviour of the M field inside the vortex core is the same as that outside the core. The only additional piece in M is a radial dependence $M_c(r)$. The matrix $M_c(r)$ is a multiple of the identity and can be moved through the matrix $M_1(\theta)$. There is also an additional piece for the gauge field, $A_{\theta c}$, which also contributes to the energy density of the vortex. Substituting the above equations in the field equations we get the following differential equations for $A_{\theta c}$ and $M_c(r)$.

$$-\frac{d^2 M_c(r)}{dr^2} - \frac{dM_c(r)}{r\ dr} = g^2 (A_{\theta c})^2(r)([T^3,[T^3,M(0)]]M(0)^{-1}) - \frac{d\bar{V}(M)}{dM}|_{M=M(0)} M(0)^{-1}$$
$$\frac{d^2 A^3_{\theta c}}{dr^2} + \frac{d}{dr}(\frac{A^3_{\theta c}}{r}) = -g M^2_c(r) A^3_{\theta c}(r) tr([T^3,M(0)]^t [T^3,M(0)]) \quad .$$
$$\frac{d\bar{V}(M)}{dM}|_{M=M(0)} M(0)^{-1} = \lambda (M_c(r)^2 - I) \tag{19}$$

The above differential equations have the same form as those of the A-N-O vortex (the matrices can be traced over and yield only some constant factors). Although this is not very surprising, because we are looking at abelian field configurations (only $A^3_\mu \neq 0$), it is interesting to note that they define the core of a vortex which is quite different from the A-N-O vortex. This difference arises because the abelian group $SO(2)$ is now embedded in $SO(3)$. The differential equations for the vortex core are quite complicated and cannot be solved analytically. For regions far from the center of the vortex (but still inside the core of the vortex) we can use the following approximate solutions for $M_c(r)$ and $A_{\theta c}(r)$.

$$A_{\theta c}(r) = (const)\frac{\exp(-\sqrt{8}gr)}{\sqrt{r}}$$
$$M_c(r) = 1 - \exp(-\sqrt{\lambda} r) \quad .$$

The implications of such vortex solutions for Wilson loops are quite striking. Unlike the A-N-O vortex, which always contributes a factor of $+1$ to the Wilson loop, or the monopole, which produces a phase depending on the solid angle subtended at it by the Wilson loop, the $Z(2)$ vortex produces a constant factor of -1 whenever it lies inside the minimal surface spanned by the Wilson loop. This effect can be demonstrated by looking at

$$\exp g \int_C A_\mu(x) dx^\mu \quad . \tag{20}$$

If the loop C encircles a vortex (and has a size larger than the core of the vortex) the Wilson loop is either $\exp 2\pi T^3$ (for $A_{\theta 0}$) or $\exp \pi T^3$ (for $A_{\theta 1}$). The trace of the first quantity is 3 whereas the trace of the second quantity is -1 (see Eq. 11). These vortices have a $Z(2)$ magnetic charge and, as far as the Wilson loop is concerned, only their total parity (odd or even number) is physical, their overall number being irrelevant. The energy of the vortex is finite and can be written as

$$E = E_1(\lambda)/g^2 \quad , \tag{21}$$

where $E_1(\lambda)$ is a quantity which depends on the parameters in the Higgs potential. The size of the vortex is determined by the parameter λ ($\frac{1}{\sqrt{\lambda}}$). The energy cannot be determined exactly in the absence of an exact solution inside the vortex core. We can also construct approximate multi-vortex solutions by superposing many single vortex solutions and making their mutual separations much larger than their individual sizes.

If we now consider the system in 2 Euclidean dimensions the vortex solutions have finite action and they can be used as a starting point for a semi-classical approximation. This is a well known scheme for approximating the partition function and was brought to its culmination in [8] where the three-dimensional Georgi-Glashow model was analyzed in the background of 't Hooft-Polyakov monopoles and shown to be in the confining phase. The semi-classical technique proceeds by expanding the partition function about the classical vortex solution by writing

$$A_\mu(x) = \bar{A}_\mu(x) + a_\mu(x)$$

$$M(x) = \bar{M}(x)\ m(x) \quad .$$

Here we denote $\bar{A}_\mu$ and $\bar{M}(x)$ as the classical solutions and $a_\mu(x)$ and $m(x)$ as the quantum fluctuations about the classical solution. The partition function can be expanded as

$$Z = \int DA_\mu DM(r) \exp -(S_c + \frac{\delta^2 S_c}{2\delta a_\mu(x)\delta a_\nu(y)} a_\mu(x) a_\nu(y) + \frac{\delta^2 S_c}{2\delta\bar{M}(x)\delta\bar{M}(y)} tr(m^t(x)m(y)) + \ ...) \quad . \tag{22}$$

(S_c is the action of the vortex solution) The semi-classical method is a good approximation provided we are in the weak coupling regime. If we simply consider the classical solution and ignore the quantum fluctuations the expectation value of the Wilson loop of area A becomes

$$\frac{1}{Z}\sum_{n=0}^{\infty}\sum_{n_1=0}^{n} n\ C_{n_1} \exp -(nE_1/(g^2))(-1)^{n_1}(\frac{A}{\Omega})^{n_1}(1-\frac{A}{\Omega})^{n-n_1} \quad . \tag{23}$$

In the above expression, $(-1)^{n_1}$ is the effect on the Wilson loop if it encloses n_1 vortices,$\frac{A}{\Omega}$ is the probability of having a vortex enclosed by the Wilson loop, $(1-\frac{A}{\Omega})$ is the probability of the vortex not enclosing the Wilson loop,$n\ E_1$ is the energy associated with n vortices with their mutual interactions ignored, n is the total number of vortices in the system, Ω is the spatial area of the system. The major approximation made is the neglect of vortex-vortex interactions (also called the dilute gas approximation) and the construction of multi-vortex solutions by superposing single vortex solutions with large mutual separations ($>> \frac{1}{\sqrt{\lambda}}$). The sum can be easily evaluated using the properties of the binomial expansion and it becomes

$$Lim_{n\to\infty}(1 - \frac{2A\rho}{n}\exp(-\frac{E_1}{g^2}))^n \quad , \tag{24}$$

giving an exponentially small string tension

$$\sigma = 2\rho\exp(\frac{-E_1}{g^2}) \quad . \tag{25}$$

(ρ is the density of vortices in the system). The density of vortices is proportional to $\exp -\frac{E_1(\lambda)}{g^2}$ and is very small in the weak coupling region. We are thus consistent in using the dilute vortex approximation. The calculation of the quantum fluctuations will require

the calculation of determinants of second order differential operators in the background of the vortex solution. Fluctuations of the vortex which are just translations or gauge rotations will lead to zero modes in the determinant. The zero-modes that appear in this integration will correspond to the translational(2) and gauge invariance(3) of the vortex solution. The zero modes can be handled by the collective co-ordinate method and it is well known [11] that integration over the zero modes gives an additional correction proportional to

$$g^{-n_v} \tag{26}$$

where n_v is the number of zero modes. The numerical factor that accompanies this requires a very detailed calculation of the determinant operator and will not be calculated here. We can write the result of the first-order quantum fluctuations as

$$g^{-5}\epsilon(\lambda) \quad . \tag{27}$$

$\epsilon(\lambda)$ is the result of the calculation of the determinant operator (without the zero modes) in the background of the vortex solution. The effect of this calculation is to renormaize the chemical potential of the vortices to

$$\frac{1}{g^5}\epsilon(\lambda)\exp\frac{-E_1(\lambda)}{g^2} \quad . \tag{28}$$

In the $2+1$ dimensional field theory the vortices were presented as static solutions but Lorentz boosts of these solutions are also solutions- the vortices form loops which can be thought of as the world-lines of particles carrying a $Z(2)$ magnetic charge. The energy of these loops will be proportional to their length (there will also be factors coming from the thickness, $\frac{1}{\sqrt{\lambda}}$). Such a dilute vortex gas can either form a entropy dominated vortex condensate with long vortex loops, or a energy dominated phase in which the vortex loops are very small. Now the vortex produces a negative sign for a Wilson loop provided it has a non-trivial linkage with it. Large vortex loops will be necessary if they have to produce the disordering (of of the kind just exhibited in 2 dimensions) of large Wilson loops, and

this is only possible in the phase in which the vortices condense. In the phase with small vortices large Wilson loops will not be disordered likewise and only a perimeter law will result. At weak coupling, where the semi-classical method is applicable, the vortex energy is very large and only small vortices are formed. However, at strong coupling the vortex entropy dominates and large vortices will be present.

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